

Rank as a function of measure

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Abstract

We establish certain topological properties of rank understood as a function on the set of invariant measures on a topological dynamical system. To be exact, we show that rank is of Young class LU (i.e., it is the limit of an increasing sequence of upper semicontinuous functions).

1 Introduction

Let (X, T) be a topological dynamical system, where X is a compact metric space and $T : X \rightarrow X$ is a continuous mapping. Denote by $\mathcal{M}_T(X)$ the set of all T -invariant Borel probability measures on X . It is well known that this is a nonempty, convex compact metric set (in the weak-star topology), whose extreme points are precisely the ergodic measures (the collection of which we denote by $\text{ex } \mathcal{M}_T(X)$). Moreover, $\mathcal{M}_T(X)$ is in fact a Choquet simplex, that is, every invariant measure μ admits a unique representation as an integral average of the ergodic measures (the ergodic decomposition). Thus, the system (X, T) gives rise to what we call an *assignment*, a function Ψ whose domain is a metric Choquet simplex K , and codomain is the set of measure-preserving systems identified up to isomorphism. Every such assignment is determined by its restriction to the set $\text{ex } K$ of extreme points of K ; the restriction assumes only ergodic measure-preserving systems and the entire assignment can be reconstructed from $\Psi|_{\text{ex } K}$ according to the ergodic decomposition. Two assignments, Ψ on a simplex \mathcal{P} , and Ψ' on a simplex \mathcal{P}' are said to be *equivalent* if there is an affine homeomorphism $\pi : \mathcal{P} \rightarrow \mathcal{P}'$ such that for every $p \in \mathcal{P}$, $\Psi(p)$ and $\Psi'(\pi(p))$ are isomorphic (as measure preserving systems).

In an attempt to understand the interplay between topological and measurable dynamics, one encounters the following natural problem:

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- characterize the abstract assignments that can be realized in topological dynamical systems.

At the moment the complete solution of this problem seems beyond our reach. There should exist some “continuity” or at least “measurability” obstructions, but we have at our disposal no good topological or measurable structure in the collection of classes of measure-preserving systems modulo isomorphism. Nonetheless, we can produce a number of necessary conditions by studying the behavior of some isomorphism invariants with values in more friendly spaces. For instance, it is fairly intuitive, that if we consider an isomorphism invariant in form of a real number r (for example the Kolmogorov-Sinai entropy), then $r(\Psi)$ should be measurable on the simplex K . Indeed, in any topological system the entropy function $h : \mathcal{M}_T(X) \rightarrow [0, \infty]$ is not only measurable; it is a nondecreasing limit of upper semicontinuous functions (i.e., of Young class LU), see [DS03].¹ Since (except on some domains, e.g. discrete or countable) not every nonnegative function is of class LU (suffice it to note that an LU function is Borel measurable), the entropy obstruction is nonvoid; it implies that not all possible assignments are admissible in topological systems.²

Following the same lines of investigation, in this paper we will seek an obstruction related to another real-valued (in fact integer-valued) isomorphism invariant, namely the rank (as defined by Ornstein, Rudolph and Weiss in [ORW82]). Notice that rank distinguishes systems of zero Kolmogorov-Sinai entropy, hence any obstruction that we find is complementary to the entropy obstruction.

Although its definition does not require ergodicity, rank has been studied mainly for ergodic systems. Any invariant measure on a standard space decomposes as an integral average of ergodic measures, hence it is important to understand how rank depends on convex combinations of mutually singular measures (and more general integral averages). In this aspect we will recall the result proven originally by J. King in [Kin88], that rank obeys a certain “additive rule” (it

¹ The entropy function is also affine. In [DS03] it is shown that there are no other entropy obstructions: every affine LU function $h : \text{ex } K \rightarrow [0, \infty]$ defined on any metrizable Choquet simplex can be modeled as the entropy function in a topological system.

²Interestingly, it has been proven (in [KO06] for homeomorphisms and independently in [Dow08] for continuous maps) that if the simplex K has at most countably many extreme points then every assignment Ψ on K assuming ergodic but not periodic “values” on the extreme points (and extended to the rest of the simplex by averaging) can be realized in a topological (even minimal) system. This is in no collision with the entropy obstruction; on a countable set every function is of class LU. Notice that this fact generalizes the celebrated Jewett–Krieger Theorem, which can be viewed as a special case concerning the one-point simplex.

cannot be affine as it is an integer-valued function). Next, we will give the main result of this note, that just like the entropy function, the rank function is also of Young class LU. This gives a non-trivial restriction on possible assignments on metrizable Choquet simplices having an uncountable number of extreme points with “values” being measure-preserving systems with entropy zero.

2 Preliminaries

Let (X, \mathcal{B}, μ) be a probability space. All subsets of X considered below are assumed to be measurable, and all partitions are (measurable and) finite. Given a partition \mathcal{P} of X and a subset $Y \subset X$, the symbol $\mathcal{P}|_Y$ denotes the partition $\{P \cap Y : P \in \mathcal{P}\}$ of Y .

Definition 2.1. For partitions \mathcal{P} and \mathcal{Q} we will write $\mathcal{P} \succ_\varepsilon \mathcal{Q}$ if there exists a set Y_ε of measure at least $1 - \varepsilon$ such that $\mathcal{P}|_{Y_\varepsilon} \succ \mathcal{Q}|_{Y_\varepsilon}$ (i.e., \mathcal{P} refines \mathcal{Q} relatively on Y_ε).

Throughout the rest of this section X denotes a separable metric space. Most of the definitions and statements hold for more general topological spaces, but for the sake of simplicity, the theory is not developed in wider generality. Exceptionally, the space X from this part of preliminaries will be later interpreted as either $\mathcal{M}_T(Y)$ or $\text{ex}\mathcal{M}_T(Y)$ where Y is a topological space.

Definition 2.2. A function $f : X \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is called *upper semicontinuous* (u.s.c.) if for all $t \in \mathbb{R}$ the sets $\{x \in X : f(x) < t\}$ are open.³

Remark 2.3. The characteristic function of a closed set is upper semicontinuous (see Chapter IV, §6.2 of [Bou98]). A function f is upper semicontinuous if and only if it is the pointwise limit of a nonincreasing sequence of continuous functions from X to $\mathbb{R} \cup \{-\infty, \infty\}$ (see p. 51 of [Roy88]).

The next definition originated in [You11]:

Definition 2.4. A function $f : X \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is of Young class LU (an LU function for short) if it is the pointwise limit of a nondecreasing sequence f_n of upper semicontinuous functions.

The reader will easily verify that the class LU is closed under finite sums, finite infima and countable suprema.

³ One is accustomed to *real-valued* u.s.c. functions, and such are always bounded from above. In our setup a u.s.c. function is either bounded from above or it assumes infinity as a value (necessarily on a closed set).

3 Rank of measure preserving systems

Let (X, \mathcal{B}, μ, T) be a measure-preserving dynamical system on a standard probability space⁴ (T is not assumed to be invertible). We will be studying the properties of an isomorphism invariant called rank. The notion was defined in [ORW82]. In fact the class of rank-one systems was known much earlier (since the 1960's) but the term “rank” was not yet used. The reader is referred also to the survey by Ferenczi ([Fer97]) for more details. Below we provide the definition of rank preceded by an auxiliary notion.

Definition 3.1. Let (X, \mathcal{B}, μ, T) be a measure-preserving dynamical system. Let $n_1, \dots, n_k \in \mathbb{N}$. We call $\mathcal{T}_1 = \{B_1, TB_1, \dots, T^{n_1-1}B_1\}$, $\mathcal{T}_2 = \{B_2, TB_2, \dots, T^{n_2-1}B_2\}, \dots, \mathcal{T}_k = \{B_k, TB_k, \dots, T^{n_k-1}B_k\}$ *k disjoint measurable towers*, if all the sets $T^i B_l \subset X$ appearing in the towers are measurable and

$$T^i B_l \cap T^{i'} B_{l'} = \emptyset \text{ for } (l, i) \neq (l', i').$$

Introduce the index set $I = I(\mathcal{T}_1, \dots, \mathcal{T}_k) \triangleq \bigcup_{l=1}^k (\{l\} \times \{0, \dots, n_l-1\})$. A *k-tower partition* \mathcal{P} associated with $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k$ is the partition of X consisting of the sets $T^i B_l$ with $(l, i) \in I$ and $R \triangleq X \setminus \bigcup_{(l,i) \in I} T^i B_l$. The sets B_l , $l = 1, \dots, k$ will be referred to as the *bases of the towers*, the sets $T^i B_l$, $(l, i) \in I$ (including the bases) *the level sets*, and R will be referred to as the *the remainder*. One writes

$$\mathcal{P} = \{T^i B_l, R\}_{(l,i) \in I}$$

We will skip the ranges of the indices (i.e. the notation $(l, i) \in I$) whenever possible.

Definition 3.2. Let $k \in \mathbb{N}$. One says that $\text{rank}(\mu) \leq k$ if for any finite partition \mathcal{Q} of X and any $\varepsilon > 0$ there exists a *k-tower partition* \mathcal{P} , such $\mathcal{P} \succ_\varepsilon \mathcal{Q}$ and the measure of the remainder of \mathcal{P} is less than ε .⁵ Otherwise we say that $\text{rank}(\mu) > k$. $\text{rank}(\mu) = k$ means that $\text{rank}(\mu) \leq k$ and $\text{rank}(\mu) > k-1$. Finally, $\text{rank}(\mu) = \infty$ if $\text{rank}(\mu) > k$ for any natural k . $\text{rank}(\mu)$ is referred to as the *rank* of (X, \mathcal{B}, μ, T) .

⁴Recall that a standard probability space is isomorphic to a union between a countable atomic measure and the unit interval equipped with the Lebesgue measure defined on the completion of the Borel σ -algebra.

⁵ The requirement on the remainder is meant to assure that every fixpoint of positive measure eventually requires a separate tower. Without this condition such a fixpoint could be included in all remainders and the resulting “rank” would be lowered by 1. The condition is automatically fulfilled for nonatomic measures, and can be dropped in any systems possessing no fixpoints of positive measure. It seems that this detail was overlooked in [Kin88].

Notice that the definition does not require μ to be ergodic. If μ is not ergodic then, since (X, \mathcal{B}, μ) is standard, we have the following formula (called the *ergodic decomposition*):

$$\mu = \int \nu_y d\xi_\mu(y),$$

where ξ_μ is the projection of μ onto the σ -algebra Σ of invariant sets, and the measures ν_y are ergodic and supported by the atoms y of Σ . Distinct ergodic measures are mutually singular.

Remark 3.3. It is well known (for ergodic systems, but easily extends to nonergodic systems as well due to the additivity of rank) that finite rank implies zero Kolmogorov–Sinai entropy (see proposition 1.8 of [Kin88]), which further implies that the transformation T is invertible μ -almost everywhere. We will use this fact several times.

We now prove the following “additive rule”, which is a simple generalization of Lemma 1.2 of [Kin88]:

Theorem 3.4. *The rank function satisfies*

$$\text{rank}(\mu) = \sum_{\nu \in \text{supp}(\xi_\mu)} \text{rank}(\nu).$$

In other words, $\text{rank}(\mu)$ can be finite only for measures which are convex combinations of finitely many ergodic measures, and then it equals the sum of ranks of the ergodic components (of course only those with strictly positive coefficients). Other measures have infinite rank.

Proof. If μ is a finite combination of ergodic measures, this follows directly from lemma 3.5 below. Every other measure can be represented as a convex combination of arbitrarily (still finitely) many mutually singular measures (each of rank at least 1 – there is no rank zero), so, by the same theorem its rank is infinite. Clearly, the sum on the right is, for such measures, also infinite, so the equality holds. ■

The following lemma is a reformulation of Lemma 1.2 of [Kin88]⁶:

Lemma 3.5. *Let (X, \mathcal{B}, μ, T) be a dynamical system. Suppose that $\mu = p\mu_1 + q\mu_2$, where $p \in (0, 1)$, $q = 1 - p$, μ_1 and μ_2 are mutually singular T -invariant measures on \mathcal{B} . Then:*

$$\text{rank}(\mu) = \text{rank}(\mu_1) + \text{rank}(\mu_2).$$

⁶In the statement of this lemma it is assumed that the measures forming the convex combination are ergodic, but this property is not actually used in the proof.

4 Rank in topological systems

Throughout, by a *topological dynamical system* (t.d.s.) we will mean a pair (X, T) , where X is a metric space (with the metric denoted by d) and $T : X \rightarrow X$ is continuous (not necessarily invertible). We will denote by $\mathcal{M}_T(X)$ the set of all T -invariant Borel probability measures on X and by $\text{ex}\mathcal{M}_T(X) \subset \mathcal{M}_T(X)$ the set of ergodic measures. It is well known that both sets are nonempty and the former set equals the simplex whose extreme points constitute the latter set. For $\mu \in \mathcal{M}_T(X)$ we obtain a measure-theoretic dynamical system $(X, \mathcal{B}_\mu, \mu, T)$, where \mathcal{B}_μ denotes the σ -algebra of Borel sets completed with respect to μ . Note that $(X, \mathcal{B}_\mu, \mu)$ is a standard probability space. We are now going to investigate the *rank function* $\text{rank} : \mathcal{M}_T(X) \rightarrow \mathbb{N} \cup \{\infty\}$, $\mu \mapsto \text{rank}(\mu)$ (computed in the system $(X, \mathcal{B}_\mu, \mu, T)$). We would like to use the “additive rule” of Lemma 3.5 (also in the form of Theorem 3.4) in this context. Notice however that for two distinct $\mu_1, \mu_2 \in \mathcal{M}_T(X)$, it may happen that $\mathcal{B}_{\mu_1} \neq \mathcal{B}_{\mu_2}$, so technically the assumption of this theorem may not hold. The following remark explains why still we can use Lemma 3.5.

Remark 4.1. For a t.d.s. rank does not depend on taking the completion. Indeed if the rank computed for the completed Borel σ -algebra is infinite then clearly it is also infinite for the Borel σ -algebra. Otherwise the map is invertible mod μ and then any tower is equal mod μ to a tower with Borel level sets. It suffices to replace the base by its subset (of equal measure) of type F_σ (a countable union of closed sets), and note that type F_σ is preserved by forward images of continuous maps on compact spaces. By invertibility, the forward images of the discarded part of the base have measure zero and can be discarded from the level sets.

As a consequence of the previous remark we will write (X, \mathcal{B}, μ, T) instead of $(X, \mathcal{B}_\mu, \mu, T)$ when no confusion arises. The main goal of this section is proving the following theorem:

Theorem 4.2. *In any topological dynamical system (X, T) the rank function $\text{rank} : \mathcal{M}_T(X) \rightarrow \mathbb{N} \cup \{\infty\}$ is of Young class LU .*

In order to prove the theorem we define an approximate notion of rank applicable to measure-theoretic dynamical systems (X, \mathcal{B}, μ, T) arising from a t.d.s. (X, T) (where X is a metric space).

Definition 4.3. Let (X, T) be a t.d.s. and let $k \in \mathbb{N}$. One says that $\text{rank}_\varepsilon(\mu) \leq k$ if there exist a measurable k -tower partition \mathcal{P} of (X, \mathcal{B}, μ, T) whose remainder satisfies $\mu(R) < \varepsilon$, and a measurable set

X_ε such that $\mu(X_\varepsilon) > 1 - \varepsilon$ and all elements of $\mathcal{P}|_{X_\varepsilon}$ have diameters smaller than ε . Otherwise we say that $\text{rank}_\varepsilon(\mu) > k$. $\text{rank}_\varepsilon(\mu) = k$ if $\text{rank}_\varepsilon(\mu) \leq k$ and $\text{rank}_\varepsilon(\mu) > k - 1$. $\text{rank}_\varepsilon(\mu) = \infty$ if $\text{rank}_\varepsilon(\mu) > k$ for all natural numbers k . $\text{rank}_\varepsilon(\mu)$ is referred to as the ε -rank of (X, \mathcal{B}, μ, T) .

Remark 4.4. The definition does not imply existence of a k -tower partition whose all sets except the remainder (of small measure) have diameters smaller than ε . To achieve small diameters, we may need to discard large parts (in relative measure) from some level sets scattered along the towers. Although the discarded parts have small measure, this may destroy the tower structure on a set of large measure.

Observe that if $\varepsilon < \varepsilon'$ then $\text{rank}_\varepsilon \geq \text{rank}_{\varepsilon'}$.

Lemma 4.5. *Let (X, T) be a topological dynamical system. Let $\mu \in \mathcal{M}_T(X)$. Then $\text{rank}(\mu) \leq k$ if and only if $\text{rank}_\varepsilon(\mu) \leq k$ for all $\varepsilon > 0$. In other words (by monotonicity)*

$$\text{rank}(\mu) = \lim_m \uparrow \text{rank}_{\varepsilon_m}(\mu),$$

whenever $\varepsilon_m \searrow 0$.

Proof. Assume that $\text{rank}_\varepsilon(\mu) \leq k$ for all $\varepsilon > 0$. Fix a partition $\mathcal{Q} = \{Q_1, \dots, Q_n\}$ and $\varepsilon > 0$. The measure μ (being a Borel measure on a metric space) is regular, therefore there exist compact sets $K_i \subset Q_i$ such that $\mu(Q_i \setminus K_i) < \frac{\varepsilon}{4n}$, $i = 1, \dots, n$. μ is also continuous from above, so there exists a $\delta > 0$ such that $\mu(K_i^\delta) < \mu(K_i) + \frac{\varepsilon}{4n}$, where K_i^δ denotes the δ -neighborhood of K_i . We can also assume δ to be so small that the sets K_i^δ are disjoint.

Let $\eta < \min\{\delta, \frac{\varepsilon}{4n}\}$ and let \mathcal{P}_η and X_η be as in the definition of $\text{rank}_\eta(\mu)$. Set

$$C_i = \bigcup \{P \cap X_\eta : P \in \mathcal{P}_\eta, (P \cap X_\eta) \cap K_i \neq \emptyset\}.$$

Observe that $K_i \cap X_\eta \subset C_i \subset K_i^\delta$. Since $\mu(X_\eta) > 1 - \eta > 1 - \frac{\varepsilon}{4n}$, we see that $\mu(C_i \triangle K_i) < \frac{2\varepsilon}{4n}$. The triangle inequality for the metric $\mu(\cdot \triangle \cdot)$ yields the estimate $\mu(C_i \triangle Q_i) < \frac{3\varepsilon}{4n}$. The set

$$A_i = \bigcup \{P \in \mathcal{P}_\eta : (P \cap X_\eta) \cap K_i \neq \emptyset\}$$

is a sum of elements of \mathcal{P}_η , we also know that $A_i \cap X_\eta = C_i$. Therefore A_i differs from C_i (in measure) by at most $\frac{\varepsilon}{4n}$, and thus $\mu(A_i \triangle Q_i) < \frac{\varepsilon}{n}$. Furthermore, the sets A_i are disjoint, which implies that $\mathcal{P}_\eta \succ_\varepsilon \mathcal{Q}$. The remainder of \mathcal{P}_η has measure less than $\frac{\varepsilon}{4n} < \varepsilon$, so we conclude that $\text{rank}(\mu) \leq k$.

The reversed implication will be first handled for nonatomic measures μ . Let $\varepsilon > 0$. Since μ is nonatomic, there exists a partition $\mathcal{Q} = \{Q_1, \dots, Q_l\}$ of X whose elements have measures smaller than ε and, moreover, diameters smaller than ε . Since $\text{rank}(\mu) \leq k$, we can find a k -tower partition \mathcal{P} such that $\mathcal{P} \succ_\varepsilon \mathcal{Q}$ and the remainder of \mathcal{P} has measure less than ε . By definition this determines a set Y_ε with $\mu(Y_\varepsilon) > 1 - \varepsilon$ and such that $\mathcal{P}|_{Y_\varepsilon} \succ \mathcal{Q}|_{Y_\varepsilon}$. Thus the elements of $\mathcal{P}|_{Y_\varepsilon}$ have diameters smaller than ε . According to Definition 4.3, we have shown that $\text{rank}_\varepsilon(\mu) \leq k$.

If μ has atoms then $\mu = p\mu' + \sum_{i=1}^n q_i\mu_i$ ($0 \leq p < 1$, $p + \sum_{i=1}^n q_i = 1$, $n \in \mathbb{N}$), where μ' is nonatomic and each μ_i is a periodic measure supported by an individual periodic orbit. Since the measures $\mu', \mu_1, \mu_2, \dots, \mu_n$ are mutually singular and the rank of any measure is at least 1, we conclude (using the “additive rule” of Lemma 3.5) that $k \geq k' + n$, where $k' = \text{rank}(\mu')$ (this also explains why the number n of periodic orbits must be finite). Given $\varepsilon > 0$, let \mathcal{P} be the $(k' + n)$ -tower partition consisting of the n periodic orbits (each viewed as a tower with singleton level sets) and a k' -tower partition of the rest of the space, satisfying the conditions of Definition 4.3 for μ' (with some set X'_ε), whose existence is established in the preceding paragraph. It is clear that \mathcal{P} fulfills the requirements of Definition 4.3 showing that $\text{rank}_\varepsilon(\mu) \leq k' + n \leq k$; the set X_ε equals the union of X'_ε and the periodic orbits (then $\mu(X_\varepsilon) > 1 - p\varepsilon > 1 - \varepsilon$). ■

The following theorem is the key observation of this work. For easier proof we assume invertibility of the transformation. Subsequently we will use it to prove Theorem 4.2 (which holds in the general case, i.e. also for non-invertible transformations).

Theorem 4.6. *Let (X, T) be an invertible (i.e., in which T is a homeomorphism) topological dynamical system. For any $\varepsilon > 0$ the function $\text{rank}_\varepsilon(\mu) : \mathcal{M}_T(X) \rightarrow \mathbb{N} \cup \{\infty\}$ is upper semicontinuous.*

Proof. We need to show that for each $t \in \mathbb{R}$, $\text{rank}_\varepsilon(\mu) < t$ holds on an open set of invariant measures. Since rank_ε assumes only natural values (or ∞) this set of measures is nonempty only for $t > 1$ and then the condition $\text{rank}_\varepsilon(\mu) < t$ can be equivalently replaced by $\text{rank}_\varepsilon(\mu) \leq k$ for some $k \in \mathbb{N}$.

Assume $\text{rank}_\varepsilon(\mu) \leq k$. This means there exists a measurable k -tower partition $\mathcal{P} = \{T^i B_l, R\}_{(l,i) \in I}$ with $\mu(R) < \varepsilon$ and a set X_ε with $\mu(X_\varepsilon) > 1 - \varepsilon$ such that $\mathcal{P}|_{X_\varepsilon}$ consists of sets with diameters smaller than ε . We will now explain why we can assume that all level sets of the towers are closed. Choose a positive number ξ such that $\mu(R) + \xi < \varepsilon$ and $\mu(X_\varepsilon) - \xi > 1 - \varepsilon$. By regularity we can find closed subsets

$B'_l \subset B_l$ so that $\mu(B_l \setminus B'_l) < \frac{\xi}{kn_l}$. Then, for all pairs $(l, i) \in I$ the images $T^i B'_l$ are closed, contained in $T^i B_l$ and

$$\mu(T^i B_l \setminus T^i B'_l) < \frac{\xi}{kn_l}$$

(here we use the assumption that T is invertible). Let $\mathcal{P}' = \{T^i B'_l, R'\}_{(l,i) \in I}$ be the k -tower partition associated with the new (smaller) bases B'_l and a new (larger) remainder set R' . The difference $R' \setminus R$ equals the union of the parts discarded from the level sets, so its measure is smaller than ξ . Thus $\mu(R') \leq \mu(R) + \xi < \varepsilon$. Similarly, the set $X'_\varepsilon = X_\varepsilon \setminus (R' \setminus R)$ has measure larger than $\mu(X_\varepsilon) - \xi > 1 - \varepsilon$. Because X'_ε differs from X_ε only within the new remainder, the partition $\mathcal{P}'|_{X'_\varepsilon}$ consists of the sets $T^i B'_l \cap X'_\varepsilon = T^i B'_l \cap X_\varepsilon \subset T^i B_l \cap X_\varepsilon$ (which have diameters smaller than ε) and $R' \cap X'_\varepsilon = R \cap X_\varepsilon$ (also of diameter smaller than ε).

From now on we assume that the original k -tower partition $\mathcal{P} = \{T^i B_l, R\}$ has closed level sets. We are going to modify the tower and the set X_ε once more, so that R becomes closed and X_ε open. Once again, choose a positive ξ such that $\mu(X_\varepsilon) - \xi > 1 - \varepsilon$ (this time the other condition, involving the remainder, will not be needed). Define

$$R^{-\delta} = \{x \in R : d(x, X \setminus R) \geq \delta\}.$$

Find δ such that $\mu(R \setminus R^{-\delta}) < \xi$ and let α denote a positive number smaller than half of the smallest distance between two distinct closed sets from the family $\{T^i B_l, R^{-\delta}\}$ (clearly $\alpha \leq \frac{\delta}{2}$). Let $\beta > 0$ be so small that $d(x, y) < \beta \implies d(T^i x, T^i y) < \alpha$ for all $0 \leq i < \max\{n_1, \dots, n_k\}$ (clearly $\beta \leq \alpha$). Define $B'_l = B_l^\beta$ (i.e., the open β -neighborhood around B_l). For every pair $(l, i) \in I$ we have $T^i B'_l \subset (T^i B_l)^\alpha$ which implies that the sets $T^i B'_l$ are pairwise disjoint, hence form a new k -tower partition \mathcal{P}' with a new smaller and closed remainder R' .

Let γ be such that $(T^i B_l)^\gamma \subset T^i B'_l$ for all pairs $(l, i) \in I$ (here we use again that T is a homeomorphism, so the new level sets $T^i B'_l$ are all open neighborhoods of the old closed level sets $T^i B_l$). Clearly, $\gamma \leq \beta \leq \alpha$. We can choose γ also smaller than half of the difference between ε and the largest diameter of an element of $\mathcal{P}|_{X_\varepsilon}$. We can now define the modified *open* version of X_ε :

$$X'_\varepsilon = \bigcup_{(l,i) \in I} (T^i B_l \cap X_\varepsilon)^\gamma \cup (R^{-\delta} \cap X_\varepsilon)^\gamma.$$

Notice that X'_ε contains X_ε except its part contained in $R \setminus R^{-\delta}$ (this is seen even if we disregard the γ -neighborhoods). So the measure of X'_ε has dropped by at most ξ and thus is still larger than $1 - \varepsilon$.

Since $(T^i B_l \cap X_\varepsilon)^\gamma \subset (T^i B_l)^\gamma \subset T^i B'_l \subset (T^i B_l)^\alpha$ for all pairs (l, i) and $(R^{-\delta} \cap X_\varepsilon)^\gamma \subset (R^{-\delta})^\alpha$, the items of the union defining X'_ε are pairwise disjoint (as $\{(T^i B_l)^\alpha, R^\alpha\}$ are pairwise disjoint), and each new level set $T^i B'_l$ intersects only one of them, namely $(T^i B_l \cap X_\varepsilon)^\gamma$. Moreover, as $T^i B'_l \subset (T^i B_l)^\alpha$ we have in fact the following equality :

$$T^i B'_l \cap X'_\varepsilon = (T^i B_l \cap X_\varepsilon)^\gamma.$$

This implies that the last item $(R^{-\delta} \cap X_\varepsilon)^\gamma$ equals the intersection of X'_ε with the remainder R' of the new tower. We have just proven that the items of the union defining X'_ε correspond to the elements of the partition $\mathcal{P}'|_{X'_\varepsilon}$. As γ is smaller than half of the difference between ε and the largest diameter of an element of $\mathcal{P}|_{X_\varepsilon}$ (and since $R^{-\delta} \subset R$), the diameters of all these items are smaller than ε .

To summarize, we have shown that if $\text{rank}_\varepsilon(\mu) \leq k$ then we can arrange the partition \mathcal{P} with a closed reminder R , so that the conditions in Definition 4.3 are fulfilled with an open set X_ε . Because the measure of a closed set is an upper semicontinuous function of the measure (see Remark 2.3), we have $\mu'(R) < \varepsilon$ and $\mu'(X \setminus X_\varepsilon) < \varepsilon$ (i.e., $\mu'(X_\varepsilon) > 1 - \varepsilon$) on an open set of measures (containing μ). The same partition \mathcal{P} and the same set X_ε now give that $\text{rank}_\varepsilon(\mu') \leq k$ for all these measures, concluding the proof. ■

We can now prove the main result of this paper.

Proof. [Proof of Theorem 4.2] If T is a homeomorphism, the result is a direct consequence of the preceding Theorem 4.6 and Lemma 4.5. For noninvertible maps we first embed (X, T) in another t.d.s. (X', T') such that T' is surjective on X' (see the paragraph after Definition 6.8.10 on p. 189 of [Dow11]). Next we construct an extension $(X'', T'') \rightarrow (X', T')$ so that T'' is a homeomorphism and every T' -invariant measure $\mu' \in M_{T'}(X')$ lifts to a unique T'' -invariant measure $\mu'' \in M_{T''}(X'')$ such that the measure-theoretic system $(X'', \mathcal{B}_{X''}, \mu'', T'')$ is isomorphic to the measure-theoretic natural extension of $(X', \mathcal{B}_{X'}, \mu', T')$. Moreover, the correspondence $\mu' \mapsto \mu''$ is a homeomorphism between $\mathcal{M}_{T'}(X')$ and $\mathcal{M}_{T''}(X'')$ (the details of this construction, the so called *topological natural extension*, can be found in [Dow11] pages 189-190 and pages 111-112). We will argue that $\text{rank}(\mu') = \text{rank}(\mu'')$. If μ' has entropy zero then T' is invertible modulo μ' , and then the system $(X', \mathcal{B}_{X'}, \mu', T')$ is isomorphic to its own natural extension, and thus to $(X'', \mathcal{B}_{X''}, \mu'', T'')$, which obviously implies the desired equality of the ranks. Otherwise both μ' and μ'' have nonzero entropy hence infinite rank. Since we already know that the rank function is of class LU on $\mathcal{M}_{T''}(X'')$, it follows that it is of the same class on $\mathcal{M}_{T'}(X')$ and hence, by restriction, on $\mathcal{M}_T(X)$. ■

5 Open Questions

We have obtained that for any topological dynamical system, the rank function defined on $\mathcal{M}_T(X)$ is of Young class LU and obeys the “additive rule” of Theorem 3.4. In particular, this function is completely determined by its restriction to $\text{ex}\mathcal{M}_T(X)$, which obviously is also of class LU. Two natural questions arise:

1. Given a metrizable Choquet simplex K and an LU function on $\text{ex} K$, is its extension to all of K by the “additive rule” automatically of class LU?
2. Are these the only “rank obstructions”? I.e., given an LU function on a metrizable Choquet simplex $r : K \rightarrow \mathbb{N} \cup \{\infty\}$ which obeys the “additive rule”, does there exist a topological dynamical system (perhaps minimal) realizing r as the rank function on the simplex of invariant measures?

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